Frames of permuting equivalences

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Dedicated to the memory of András Huhn

András Huhn established frames as the fundamental tool in the equational theory of modular lattices. In the present note we use this algebraic point of view for an easy approach to von Neumann's Coordinatization Theorem [11] — completing a program of FRINK [5] and JÓNSSON [9] based on the abelian group representation given by the Embedding Theorem.

This approach can be extended to permuting equivalence representations of lattices with spanning frames of order $n \geqq 3$. The loop associated with the net provides a module representation for the sublattice generated by the frame and its coordinate ring. From this we can derive a lattice identity separating lattices of permuting equivalences on finite sets and finite lattices having an (infinite) permuting equivalence representation.

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1. The associated group

A *frame* Φ of order n in a lattice L consists of elements a_i, c_{ij} $(1 \leqq i \neq j \leqq n)$ such that

$$a_i \cap \sum_{k \neq i} a_k = \bigcap_k a_k = a_i \cap c_{ij}, \quad a_i + c_{ij} = a_i + a_j,$$

$$c_{ik} = c_{ki} = (c_{ij} + c_{jk}) \cap (a_i + a_k).$$

It is *spanning* if $\bigcap a_k$ and $\sum a_k$ are the bounds of L. For every module A we have the *canonical* frame in the lattice $L(A^n)$ of all submodules of A^n given by

$$\{(0, \ldots, \underset{i}{x}, \ldots, 0) \mid x \in A\}, \quad \{(0, \ldots, \underset{i}{x}, \ldots 0, \ldots, -\underset{j}{x}, \ldots, 0) \mid x \in A\}.$$

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The *coordinate domain* R_{ij} of Φ in L consists of all r in L such that

$$r+a_j = a_i+a_j, \quad r\cap a_j = a_i\cap a_j,$$
$$r = (r+a_k)\cap(a_i+a_j) = (r+c_{ik})\cap(a_i+a_j) = (r+c_{jk})\cap(a_i+a_j)$$

for all $k\neq i,j$. For modular L the last identity is superfluous.

A frame Φ contained in the lattice $\Pi(E)$ of all partitions on the set E is called *permuting*, if any two of the a_i's permute and c_{ij} with a_i for all i, j (it suffices to consider a spanning tree of pairs $\{i,j\}$ for which c_{ij} and c_{jk} permute, too). Then, r is in R_{ij} if it permutes with all a_k, c_{ik}, and c_{jk} $(k\neq i,j)$ and is a complement of a_j in $[a_ia_j,\ a_i+a_j]$.

Often, we prefer to think of equivalence relations. In particular, with any subgroup B of an abelian group A we associate the congruence relation β on A given by $x\beta y$ iff $x-y\in B$. A closer look at the loop associated with a net yields

Theorem 1. *Let Φ be a permuting spanning frame of order $n\geqq 3$ in $\Pi(E)$. Then there is an abelian group A and a bijection $\varphi\colon E\to A^n$ mapping Φ onto congruences associated with the canonical frame of A^n and all coordinate domain elements onto congruences of A^n.*

Corollary 2. *A permuting frame of order $n\geqq 3$ in a partition lattice generates, together with all its coordinate domains, a complete sublattice of permuting equivalences.*

Proof. Denote the frame by α_i and ε_{ij}. In view of the permutability and independence of the α_i we may assume $E=A_1\times\ldots\times A_n$ with $(a_1, \ldots, a_n)\ \alpha_i\ (b_1, \ldots, b_n)$ iff $a_j=b_j$ for all $j\neq i$. Choose an element 0_i in A_i for each i. Then

$$f_{ij}(x) = y \quad\text{iff}\quad (0, \ldots, \underset{i}{x}, \ldots, 0)\ \varepsilon_{ij}\ (0, \ldots, \underset{j}{y}, \ldots, 0)$$

defines a bijection of A_i onto A_j mapping 0_i onto 0_j — due to the permutability of ε_{ij} with α_i and α_j. The normalization condition for the frame yields $f_{jk}\circ f_{ij}=f_{ik}$. Thus, we may identify A_i with A_j via f_{ij} to obtain $E=A^n$ with

$$(a_1, \ldots, a_n)\ \varepsilon_{ij}\ (b_1, \ldots, b_n) \quad\text{iff}\quad a_i = b_j \quad\text{and}\quad a_k = b_k \quad\text{for all}\quad k\neq i,j$$

provided that $a_j=0=b_i$. Namely, let $i=1$, $j=2$. Since $\varepsilon_{12}\subseteq\alpha_1+\alpha_2$ we may assume $a_k=b_k$ for all $k\geqq 3$. If these are 0 the claim is obvious. The general case reduces to this one since

$$a\,\varepsilon_{12}\,b \quad\text{iff}\quad (a_1, 0, \ldots, 0) \quad \varepsilon_{12}+\sum_{k\geqq 3}\alpha_k \quad (0, b_2, 0, \ldots, 0)$$

in view of $(\alpha_1+\alpha_2)\cap(\varepsilon_{12}+\sum_{k\geqq 3}\alpha_k)=\varepsilon_{12}$.

The 3-net (cf. Denes and Keedwell [3]) $\alpha_i, \alpha_j, \varepsilon_{ij}$ on the $\alpha_i+\alpha_j$-class of $(0, \ldots, 0)$ yields for every $i\neq j$ and a, b in A a uniquely determined $c=a+_{ij}b$ in A such that

$$(0, \ldots, \underset{i}{a}, \ldots, 0, \ldots, \underset{j}{b}, \ldots, 0)\ \varepsilon_{ij}\ (0, \ldots, \underset{i}{c}, \ldots, 0).$$

Claim. A with 0 and $+_{ij}$ is an abelian group not depending on i, j.

We may assume $n=3$, $i\neq j\neq k\neq i$, e.g. $i=1$, $j=2$, $k=3$. Observe that

$$\alpha_i\cap(\varepsilon_{ij}+\varepsilon_{jk}+\varepsilon_{ik}) = \mathrm{id}$$

due to the frame relations. We have

$$a+_{ij}b = b+_{ji}a$$

since $(a+_{12}b, 0, 0)\,\varepsilon_{12}\,(a, b, 0)\,\varepsilon_{21}\,(0, b+_{21}a, 0)\,\varepsilon_{12}\,(b+_{21}a, 0, 0)$, and

$$a+_{ij}b = a+_{ik}b$$

since $(a+_{12}b, 0, 0)\,\varepsilon_{12}\,(a, b, 0)\,\varepsilon_{23}\,(a, 0, b)\,\varepsilon_{13}\,(a+_{13}b, 0, 0)$. Now, indepedence and commutativity follow. 0 is neutral, obviously. Due to the permutability of α_2 and ε_{12} for any a there is (b, c, d) with $(0, 0, 0)\ \varepsilon_{12}\ (b, c, d)\ \alpha_2\ (a, 0, 0)$, whence $b=a$, $d=0$, and $a+c=0$. Finally, associativity follows from

$$\big((a+b)+c, 0, 0\big)\,\varepsilon_{13}\,(a+b, 0, c)\,\varepsilon_{12}\,(a, b, c)\,\varepsilon_{23}\,(a, 0, b+c)\,\varepsilon_{13}\,\big(a+(b+c), 0, 0\big).$$

For ϱ in R_{12} we have $\varrho\subseteq\alpha_1+\alpha_2$ whence

$$(x, y, c)\ \varrho\ (a, b, c) \quad \text{iff} \quad (x, y, d)\ \varrho\ (a, b, d)$$

since $(x, y, c)\ \varrho\ (a, b, c)$ implies $(x, y, d)\ \alpha_3\ (x, y, c)\ \varrho\ (a, b, c)\ \alpha_3\ (a, b, d)$ and since $\varrho=(\varrho+\alpha_3)\cap(\alpha_1+\alpha_2)$. Also,

$$(x, y, 0)\ \varrho\ (a, b, 0) \quad \text{implies} \quad (x+u, y, 0)\ \varrho\ (a+u, b, 0)$$

since $(x+u, y, 0)\ \ \varepsilon_{13}\ (x, y, u)\ \ \varrho\ \ (a, b, u)\ \ \varepsilon_{13}\ (a+u, b, 0)$. With the corresponding property for the second component we get that $(x, y, z)\ \varrho\ (a, b, c)$ implies $(x+u, y+v, z+w)\ \varrho\ (a+u, b+v, c+w)$ which means that ϱ is a congruence of the group A^n.

2. The associated module

When studying faithful permutable representations of a lattice with a spanning frame of order $n\geqq 3$ we may, in view of Theorem 1, restrict attention to sublattices of $L(A^n)$ with canonical Φ, A an abelian group. Then, each element of the coordinate domain R_{ij} can be considered the graph of an endomorphism $x\mapsto -xr$ of A

$$\{(0, \dots, \underset{i}{x}, \dots, 0, \dots, -\underset{j}{xr}, \dots, 0)\,|\,x\in A\} = :r_{ij},$$

$r_{ik}=(r_{ij}+r_{jk})\cap(a_i+a_k)$. With von Neumann's addition and multiplication

$$(s-r)_{ij} = \big((s_{ij}+c_{jk})\cap(a_j+r_{ik})+a_k\big)\cap(a_i+a_j),$$

$$(sr)_{ik} = (r_{ij}+s_{jk})\cap(a_i+a_k),$$

the R_{ij} are isomorphic to a subring of the endomorphism ring of A. Also, if L is generated by Φ and the R_{ij} then it is contained in the lattice $L(A_T^n)$ of right T-submodules, where T is the commuting ring of R and A, i.e. the endomorphism ring of A_R. Finally, if S is a subring of T and A_S a cyclic module then identification of A_S with S yields a representation of L in the lattice $L({}_SS^n)$ of left S-submodules of S^n.

Proposition 3. *Let R be a commutative completely primary uniserial ring, Φ the canonical frame of $L=L({}_RR^n)$, and $\varphi\colon L\to\Pi(E)$ a representation with permuting and spanning frame $\varphi(\Phi)$. Then $|E|\leqq 1$ or there is a bijection ψ of E onto a direct sum ${}_RM$ of modules ${}_RR^n$ such that $\psi\circ\varphi$ becomes the diagonal embedding of L into $L({}_RM)$.*

These lattices L are particular primary lattices of type (n) in the sense of Jónsson and Monk [10]. The proposition allows to view every permuting equivalence representation as derived from the Jónsson—Monk coordinatization.

Proof. Since L is simple and generated by $\Phi\cup R_{12}$ (cf. [6], 2.9) it suffices to consider φL as a sublattice of $L(A_R^n)$ and $\varphi\Phi$ canonical. Now, A_R is a direct sum of cyclic modules which yields a decomposition of $\varphi\Phi$ (see Herrmann and Huhn [7], Section 2) and of the coordinate domain φR_{12}. Thus, all of φL is decomposed which means that we have a direct sum of representations with cyclic A_R's. The latter are full lattices $L({}_RR^n)$ since these are generated by $\Phi\cup R_{12}$.

By the *indices* of a partition β in α we mean the numbers of β-classes in the classes of α. If α and β permute then $\alpha\cap\beta\subseteq\beta$ and $\alpha\subseteq\alpha+\beta$ have the same set of indices.

Corollary 4. *Let $L\subseteq\Pi(E)$ be a simple lattice of permuting equivalences. Then any two prime quotients have the same set of indices. It consists of powers of p if E is finite and L contains a projective plane of order p as a sublattice.*

Jónsson [8] has represented the gluing of two Arguesian projective planes of different characteristics over a 2-element interval as a lattice of permuting equivalences on an infinite set — a finite set being impossible by the above. Moreover, we have

Corollary 5. *There is a finite lattice having a permuting equivalence representation which is not contained in the variety generated by all lattices of permuting equivalences on finite sets.*

Proof. Let p and q be different primes, L and $\hat{L}$ resp. the lattice which is the union of a projective 3-space $L_p=[0, b]$ of order p and $L_q=[a, 1]$ of order q such that $ab\prec b$ and $a+b\succ a$ where $b\geqq a$ resp. $b\ngeqq a$. $\hat{L}$ is a subdirect product of L and a 2-element lattice, so it has a permuting equivalence representation this being the case for L according to JÓNSSON [8] — cf. the Appendix. On the other hand, in view of Corollary 4 we have one, hence both of $\varphi 0=\varphi b$ and $\varphi 1=\varphi a$ for every homomorphism φ of $\hat{L}$ into a lattice of permuting equivalences on a finite set. This gives rise to a separating identity since L is a projective modular lattice — as is well known.

Indeed, let φ be a homomorphism from a modular lattice M onto $\hat{L}$. Now, L_p and L_q are projective modular lattices according to FREESE [4], so we may choose preimages $L'_p \cong L_p$ and $L'_q \cong L_q$ in M and $[c')$ where c' is in L'_p with $\varphi c'=ab$. Let d' be in L'_q with $\varphi d'=a+b$, let b' be the top of L'_p and a' the bottom of L'_q. Then there is a sublattice L''_q of M mapped onto L_q, isomorphically, with bottom $a''=a'$ and $d''=b'd'+a'$ a point. Indeed, choosing a 4-frame of L'_q containing d' reduction with d'' yields a 4-frame of characteristic q generating L''_q — cf. [6], Corollary 3.4. Similarily, reduction with $b'd$ and $a'b'$ yields a 4-frame of characteristic p generating an isomorphic preimage L''_p of L_p such that $b'd/a'b'$ transposes down to b''/e'' where b'' is the top and e'' a plane of L''_p. Then, $L''_p \cup L''_q$ is a sublattice of M mapped onto L, isomorphically.

3. Von Neumann's Theorem

A ring R with 1 is regular if its principal left ideals form a complemented sublattice of the lattice of all left ideals or, equivalently, if every principal left ideal is generated by an idempotent — the same characterization is valid on the right. Equivalently, the finitely generated submodules of the left R-module ${}_RR^n$ form a complemented modular lattice $L_{fg}({}_RR^n)$ — see SKORNYAKOV [12], Chapter 2. According to von Neumann every complemented modular lattice with a spanning frame of order $n \geqq 4$ (or $n=3$ and Arguesian — JÓNSSON [9]) can be respresented in this way. But, such a lattice can be embedded into the subgroup lattice of an abelian group, firsthand, due to FRINK's Embedding Theorem [5], resp. JÓNSSON [8] — see also CRAWLEY—DILWORTH [1], Chapter 13. The frame can be chosen canonical, thus the following suffices for a proof of the Coordinatization Theorem. Of course, this approach uses the coordinatization of projective spaces.

Theorem 6. *Let A be an abelian group, L a complemented sublattice of $L(A^n)$, $n \geqq 3$, containing the canonical frame Φ with coordinate ring R. Then R is regular and*

$$\varphi(M) = \{(xr_1, \ldots, xr_n) \mid x \in A,\ (r_1, \ldots, r_n) \in M\}$$

defines an isomorphism of $L_{fg}({}_RR^n)$ onto L.

In this sense *every faithful representation of a complemented modular lattice with permuting spanning frame of order $n \geqq 3$ it obtained from the von Neumann—Jónsson coordinatization.*

Proof. The corollary follows from Theorem 1 and Claim 8 below. The proof of the theorem mimics, in the coordinatizing module, the calculations of VON NEUMANN [11] and DAY and PICKERING [2]. Write $\bar{r}$ for $(r_1, \ldots, r_n)$ and $\varphi\bar{r}$ for $\varphi(R\bar{r}) = \{(xr_1, \ldots, xr_n) \mid x \in A\}$. Let L' be the sublattice of L generated by Φ and R_{12}.

Claim 1. $\varphi(s\bar{r}) \leqq \varphi\bar{r}$, $\varphi(\bar{s}+\bar{r}) \leqq \varphi\bar{s}+\varphi\bar{r}$, so φ preserves joins.

Claim 2. $\varphi\bar{r} \in L$ if there is an i with $r_i = 0$, r_i invertible, or $r_i\bar{r} = \bar{r}$.

Proof. $\varphi\bar{r} = \bigcap_{j>1} \big((r_j)_{1j} + \sum_{k>1, k\neq j} a_k\big)$ for r_1 invertible,

$$\varphi\bar{r} = \big(\sum_{j>1} a_j\big) \cap \big(a_1 + \varphi(1, r_2, \ldots, r_n)\big) \quad \text{for} \quad r_1 = 0,$$

$$\varphi\bar{r} = \big(\varphi(r_1, 0, \ldots, 0) + \sum_{j>2} a_j\big) \cap \varphi(1, r_2, \ldots, r_n) \quad \text{for} \quad r_1\bar{r} = \bar{r}.$$

Claim 3. $R\bar{r} \subseteqq R\bar{s}$ iff $\varphi\bar{r} \subseteqq \varphi\bar{s}$ provided that $r_i = s_i = 0$ for an i.

Proof. Let $\varphi\bar{r} \subseteqq \varphi\bar{s}$ and $r_n = s_n = 0$. Then let

$$\bar{r}/\bar{s} = \{(x, y, 0, \ldots, 0) \mid x, y \in A, xr_i = -ys_i \text{ for } i < n\},$$

and note

$$\bar{r}/\bar{s} = \bigcap_{i<n} \big((r_i)_{1n} + (s_i)_{2n}\big) \cap (a_1 + a_2) \in L.$$

$\bar{r}/\bar{s} + a_2 \supseteqq a_1$ since for all x in A one has $-x\bar{r} \in \varphi\bar{r} \subseteqq \varphi\bar{s}$ which means $-x\bar{r} = y\bar{s}$ for a y in A. Now, let b be a complement of $a_2 \cap \bar{r}/\bar{s}$ in $[0, \bar{r}/\bar{s}]$. Then $b \in R_{12}$, so we have t in R with $b = t_{12} \subseteqq \bar{r}/\bar{s}$. But this implies that for every z in A there are x, y in A such that $z = x$, $-zt = y$, and $xr_i = -ys_i$ for $i < n$. Consequently, $zts_i = zr_i$ for all z and i. Since R consists of endomorphisms of A this means $ts_i = r_i$ and $t\bar{s} = \bar{r}$. So $R\bar{r} \subseteqq R\bar{s}$. The converse is clear.

Claim 4. For every $b \subseteqq a_i$ there is an idempotent r in R such that

$$b = \varphi(0, \ldots, \underset{i}{r}, \ldots, 0).$$

Proof. Let $i = 2$, d a complement of $(b + c_{12}) \cap a_1$ in $[0, a_1]$ and $e = d + (b + a_1) \cap c_{12}$. Then by modularity

$$e + a_2 \supseteqq d + (b + a_1) \cap (c_{12} + b) \supseteqq a_1,$$

$$e \cap a_2 = (b + a_1) \cap (d + c_{12}) \cap a_2 = (b + a_1 \cap a_2) \cap (d + c_{12}) = b \cap (d + c_{12}) =$$

$$= b \cap \big(d \cap (b + c_{12}) + c_{12}\big) = b \cap c_{12} = 0.$$

Therefore, $e \in R_{12}$ which means that $e = r_{12}$ for an r in R and

$$\varphi(0, r, 0, \ldots, 0) = (r_{12} + a_1) \cap a_2 = (a_1 + b) \cap a_2 = b.$$

In addition, $a_1 + c_{12} \cap e \supseteq a_1 + (b + a_1) \cap c_{12} \supseteq b + a_1 \supseteq e$, so $e = a_1 \cap e + c_{12} \cap e$, and thus

$$r_{12} = a_1 \cap r_{12} + c_{12} \cap r_{12} =$$

$$= \{(x, 0, \ldots, 0) \mid x \in A,\ xr = 0\} + \{(y, -y, 0, \ldots, 0) \mid y \in A,\ y = yr\}.$$

In other words, for every z in A there are x, y in A such that $z = x + y$, $zr = y$, $xr = 0$, and $y = yr$, whence $zrr = zr$. Thus, $rr = r$.

Claim 5. R is a regular ring.

Proof. For r in R we have by Claim 4 an idempotent e with $\varphi(r, 0, \ldots, 0) = \varphi(e, 0, \ldots, 0)$ and $Rr = Re$ by Claim 3.

The following is shown in Skornyakov [12], Chapter 2, § 5., Lemma 3.

Claim 6. For $M \subseteq R^k \times 0^{n-k}$ there is $\bar{s}$ with $s_k \bar{s} = \bar{s}$, $M = R\bar{s} + M \cap R^{k-1} \times 0^{n-k+1}$.

Claim 7. $M \subseteq R^m \times 0^{n-m}$ has generators $\bar{r}^{(1)}, \ldots, \bar{r}^{(m)}$ such that

$$r_k^{(k)} \bar{r}^{(k)} = \bar{r}^{(k)} \quad \text{and} \quad r_j^{(k)} = 0 \quad \text{for all} \quad k \leqq m, \quad k < j.$$

The proof is by induction on m using Claims 4 and 6.

Choosing such a generating set G for M we have $\varphi(M) = \sum_{\bar{r} \in G} \varphi \bar{r}$ by Claim 1. Thus by Claim 2 we have

Claim 8. $\varphi(M)$ belongs to L'.

Claim 9. Let $U \subseteq R^{k-1} \times 0^{n-k+1}$, $r_k \bar{r} = \bar{r}$, $s_k \bar{s} = \bar{s}$, and $r_j = s_j = 0$ for all $j > k$. Then $\varphi \bar{r} \subseteq \varphi \bar{s} + \varphi(U)$ implies $\bar{r} \in R\bar{s} + U$.

Proof. Since $\varphi \bar{r} \subseteq \varphi \bar{s} + \varphi(U)$, we have for all x in A elements y, z in A and $\bar{t}$ in U such that $x r_k r_i = y s_k s_i + z t_i$ for all i. Since $t_j = 0$ for $j \geqq k$ it follows $x r_k = x r_k r_k = y s_k s_k = y s_k$, $x r_k r_k = x r_k s_k$, and $x r_k (\bar{r} - \bar{s}) = z \bar{t}$. Consequently, $\varphi(r_k(\bar{r} - \bar{s})) \subseteq \varphi \bar{t}$ and $R r_k (\bar{r} - \bar{s}) \subseteq R\bar{t}$ by Claim 3, whence $R\bar{r} = R r_k \bar{r} \subseteq R r_k \bar{s} + R\bar{t} \subseteq R\bar{s} + U$.

Claim 10. $\varphi(N) \subseteq \varphi(M)$ implies $N \subseteq M$, so φ is one-to-one.

Let $M \subseteq R^k \times 0^{n-k}$ and proceed by induction on k. Let $\bar{r}^{(m)} \neq 0$ be a generator of N according to Claim 7. Then $\varphi \bar{r}^{(m)} \subseteq \varphi(N) \subseteq \varphi(M)$, $\bar{r}_j^{(m)} = 0$ for $j > m$ and $m \leqq k$. If $m = k$ choose $\bar{s}$ by Claim 6 such that $M = R\bar{s} + U$ with $U = M \cap R^{k-1} \times 0^{n-k+1}$. Then $\bar{r}^{(m)} \in M$ by Claim 9. If $m < k$ let $U = (M + 0^k \times R^{n-k}) \cap (R^{k-1} \times 0^{n-k+1})$. Then $\varphi \bar{r}^{(m)} \subseteq \varphi(U)$, $\bar{r}^{(m)} \in U$ by the inductive hypothesis, and $\bar{r}^{(m)}$ is in M since $\bar{r}_j^{(m)} = 0$ for $j \geqq k$.

Claim 11. φ is an onto map.

Proof. We show by induction on k that $a \leqq \sum_{i \leqq k} a_i$ is in the image. Let $f = \sum_{i<k} a_i$, $a \in L$, $a \leqq f + a_k$. Choose a complement b of $a+f$ in $[a, f+a_k]$ and c of $a \cap f$ in $[0, b]$. It follows

$$a \cap f + a \cap c = a \cap (a \cap f + c) = a \cap b = a,$$

$$c \cap f = c \cap b \cap (a+f) \cap f = c \cap a \cap f = 0,$$

$$c+f = c + a \cap f + f = b + f = b + a + f = f + a_k.$$

Now, $f = A^{k-1} \times 0^{n-k+1}$, $a_k = 0^{k-1} \times A \times 0^{n-k+2}$, and c is a subgroup of $f+a_k$. Thus, c defines a homomorphism of A into A^{k-1}, i.e.

$$c = \{(xr_1, \dots, xr_{k-1}, x, 0, \dots, 0)\}$$

for suitable r_i in R. Then there is a subgroup D of A such that

$$d = a \cap c = \{(xr_1, \dots, xr_{k-1}, x, 0, \dots, 0) \mid x \in D\},$$

$$(d+f) \cap a_k = \{(0, \dots, 0, \underset{k}{x}, 0, \dots, 0) \mid x \in D\} = \varphi(0, \dots, \underset{k}{s}, \dots, 0)$$

for an idempotent s in view of Claim 4. Let

$$\bar{t} = (sr_1, \dots, sr_{k-1}, s, 0, \dots, 0).$$

Then $s\bar{t} = \bar{t}$, $\varphi\bar{t} = \varphi(R\bar{t}) = d$. By the inductive hypothesis we have M' with $\varphi(M') = a \cap f$. Hence $\varphi(M' + R\bar{t}) = a$.

Addendum: Gluing of two representations

The proof of Lemma 3.5 in Jónsson [8] contains the following construction: Let L be the union of the ideal L_0 generated by b and the filter L_1 generated by a, $a \leqq b$. Let $\varphi_i \colon L_i \to \Pi(E_i)$ be representations for $i = 0, 1$, let $\alpha = \varphi_0 a$, $\beta = \varphi_1 b$. For each β-class X let a map γ_X of E_0 onto X with kernel α be given such that for all c in $[a, b]$ and x, y in E_0

$$x \ \varphi_0 c \ y \quad \text{if and only if} \quad \gamma_X x \ \varphi_1 c \ \gamma_X y.$$

Let $E = E_0 \times B$ where B is the set of β-classes. Define φc on E by

$$(x, X) \ \varphi c \ (y, Y) \ \text{ iff } \begin{cases} X = Y \ \text{ and } \ x \ \varphi_0 c \ y & \text{for } c \leqq b \\ \gamma_X x \ \varphi_1 c \ \gamma_Y y & \text{for } c \geqq a. \end{cases}$$

Lemma 7 (Jónsson [8]). *φ is a representation of L on E which is permuting if and only if both φ_0 and φ_1 are permuting.*

Corollary 8. *There is a lattice not contained in any modular congruence variety but having a permuting equivalence representation on a finite set.*

Proof. For any prime p let $L_0=L(C^3_{p^2})$ and $L_1=L({}_RR^3)$ where $R=F_p[x]/x^2$, F_p the field of order p. Let $a=0\times C^2_{p^2}$, $b=R\times 0^2$, and L the lattice obtained by gluing L and L_1 over the 3-element chains $[a, 1_{L_0}]$ and $[0_{L_1}, b]$ to get $a=0_{L_1}\prec c\prec b= =1_{L_0}$. Let $\varphi_i\colon L_i\to\Pi(E_i)$ be the canonical representations. To define the γ_X observe that E_0/α as well as each β-class has p^2 elements and is partitioned by $\varphi_i c$ into p classes of p elements. By the lemma we have a permuting equivalence representation on a p^{10}-element set. On the other hand L is not contained in any modular lattice variety generated by the congruence lattices of a class of algebraic structures closed under particular subdirect products, namely the congruences; see [13].

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